

# A MIXED $\tau$ -ELECTROPRODUCTION SUM RULE FOR $|V_{us}|$

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The interpretation of results of recent  $\tau$  decay determinations of  $|V_{us}|$ , which yield values  $\sim 3\sigma$  low compared to 3-family unitarity expectations, is complicated by the slow convergence of the relevant integrated  $D = 2$  OPE series. We introduce a class of new sum rules involving both electroproduction and  $\tau$  decay data designed to deal with this problem by strongly suppressing  $D = 2$  OPE contributions at the correlator level. Experimental complications are briefly discussed and an example of the improved control over theoretical errors presented. The uncertainty on the resulting determination,  $|V_{us}| = 0.2202(39)$ , is entirely dominated by experimental errors, and should be subject to significant near-term improvement.

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## I. INTRODUCTION

The CKM matrix element,  $|V_{us}|$ , is one of the fundamental parameters of the Standard Model (SM). Determinations from multiple sources can help to improve the accuracy with which it is known and/or test for the presence of non-SM contributions in strangeness-changing weak processes. Current analyses of  $K_{\ell 3}$  and  $\Gamma[K_{\mu 2}]/\Gamma[\pi_{\mu 2}]$  [1], using lattice input for  $f_+(0)$  and  $f_K/f_\pi$ , respectively [2, 3], yield values which are in good mutual agreement and compatible with the expectations of 3-family unitarity [4, 5, 6, 7].  $|V_{us}|$  can also be obtained using flavor-breaking (FB) hadronic- $\tau$ -decay-based sum rules [8, 9, 10, 11]. Recent  $\tau$  decay analyses [10, 11], employing updated information [12, 13, 14, 15, 16, 17, 18] on the older strange decay distribution [19, 20], yield values  $\sim 3\sigma$  below 3-family-unitarity expectations.

In existing  $\tau$  decay determinations, the dominant source of error on  $|V_{us}|$  is the uncertainties on weighted integrals over the inclusive strange decay distribution. This error will be significantly reduced by ongoing B-factory analyses. Nominal theoretical errors, estimated with conventional prescriptions, are small, holding out the prospect of results competitive with those from  $K_{\ell 3}$  and  $\Gamma[K_{\mu 2}]/\Gamma[\pi_{\mu 2}]$ , once the B-factory analyses are complete. A potential complication, however, arises from the slow convergence of the relevant integrated  $D = 2$  OPE series. Evidence suggests that theoretical errors may be underestimated (in some cases, significantly) as a consequence of this behavior.

In this paper we consider alternate sum rules for  $|V_{us}|$ , involving both  $\tau$  and electropro-

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duction, rather than just  $\tau$ , spectral data. The combinations chosen have, by construction, *already at the correlator level*, a strong suppression of the potentially problematic  $D = 2$  OPE series, and hence also a strongly reduced  $D = 2$  truncation contribution to the theoretical uncertainty. The rest of the paper is organized as follows. In Section II we first briefly outline the purely  $\tau$  decay approach and associated  $D = 2$  OPE convergence problem. Then, in Section III we introduce and discuss the alternate, mixed  $\tau$  decay-electroproduction sum rules. Finally, Section IV outlines the spectral and OPE input, discusses some experimental complications, and provides an illustration of the utility of the mixed sum rule approach.

## II. THE HADRONIC $\tau$ DECAY DETERMINATION OF $|V_{us}|$

For any correlator,  $\Pi$ , without kinematic singularities, and any analytic weight,  $w(s)$ , analyticity implies the finite energy sum rule (FESR) relation,

$$\int_0^{s_0} w(s) \rho(s) ds = -\frac{1}{2\pi i} \oint_{|s|=s_0} w(s) \Pi(s) ds, \quad (1)$$

where  $\rho(s)$  is the spectral function of  $\Pi(s)$  and the OPE expansion of  $\Pi(s)$  can be employed on the RHS for sufficiently large  $s_0$ .  $|V_{us}|$  is obtained by applying this relation to the FB correlator difference  $\Delta\Pi_\tau(s) \equiv [\Pi_{V+A;ud}^{(0+1)}(s) - \Pi_{V+A;us}^{(0+1)}(s)]$ , where  $\Pi_{V/A;ij}^{(J)}$  are the spin  $J = 0, 1$  components of the flavor  $ij$ , vector (V) or axial vector (A) current two-point functions, and the corresponding spectral functions,  $\rho_{V/A;ij}^{(J)}$ , are related to the differential distributions,  $dR_{V/A;ij}/ds$ , of the normalized flavor  $ij$  V or A current induced decay widths,  $R_{V/A;ij} \equiv \Gamma[\tau^- \rightarrow \nu_\tau \text{ hadrons}_{V/A;ij}(\gamma)]/\Gamma[\tau^- \rightarrow \nu_\tau e^- \bar{\nu}_e(\gamma)]$ , by [21]

$$\frac{dR_{V/A;ij}}{ds} = \frac{12\pi^2 |V_{ij}|^2 S_{EW}}{m_\tau^2} \left[ w_T^{(0,0)}(y_\tau) \rho_{V/A;ij}^{(0+1)}(s) - w_L^{(0,0)}(y_\tau) \rho_{V/A;ij}^{(0)}(s) \right] \quad (2)$$

with  $y_\tau = s/m_\tau^2$ ,  $w_T^{(0,0)}(y) = (1-y)^2(1+2y)$ ,  $w_L^{(0,0)}(y) = 2y(1-y)^2$ ,  $V_{ij}$  the flavor  $ij$  CKM matrix element,  $S_{EW}$  a short-distance electroweak correction [22], and  $(0+1)$  denoting the sum of  $J = 0$  and  $1$  contributions. The  $J = 0$  contribution to any  $J = 0 + 1/J = 0$  decomposition will be referred to as “longitudinal” in what follows.

The use of the  $J = 0 + 1$  difference,  $\Delta\Pi_\tau$ , is the result of the extremely bad behavior of the integrated longitudinal  $D = 2$  OPE series [23], which precludes working with FB FESRs based on the linear combination of  $J = 0, 0 + 1$  spectral functions appearing in Eq. (2). The  $\rho_{V/A;ij}^{(0+1)}(s)$ , and from these,  $\Delta\rho_\tau$  are obtained by identifying and subtracting, bin-by-bin, the longitudinal contributions to  $dR_{V/A;ij}/ds$ . This can be done with good accuracy because, apart from the  $\pi$  contribution to  $\rho_{A;ud}^{(0)}$  and  $K$  contribution to  $\rho_{A;us}^{(0)}$  (which are determined by  $f_\pi$  and  $f_K$ , respectively, and hence very accurately known) all contributions to  $\rho_{V/A;ij}^{(0)}$  are doubly chirally suppressed, by factors of  $O[(m_i \mp m_j)^2]$ . The  $ij = ud$  longitudinal contributions are thus, to high accuracy, saturated by the  $\pi$  pole

term. Continuum longitudinal  $ij = us$  contributions, which are small, but not entirely negligible, are determined from dispersive [24] and sum rule [25] analyses of the strange scalar and pseudoscalar channels, respectively, analyses which are strongly constrained by their implications for  $m_s$  [26].

Given  $w(s)$  and  $s_0 \leq m_\tau^2$ ,  $|V_{us}|$  is determined by first constructing, from the longitudinally subtracted  $dR_{V/A;ij}/ds$ , the spectral integrals

$$R_{V/A;ij}^w(s_0) \equiv 12\pi^2 S_{EW} |V_{ij}|^2 \int_0^{s_0} \frac{ds}{m_\tau^2} w(s) \rho_{V+A;ij}^{(0+1)}(s) , \quad (3)$$

and, from these, the FB combinations,

$$\delta R_{V+A}^w(s_0) = [R_{V+A;ud}^w(s_0)/|V_{ud}|^2] - [R_{V+A;us}^w(s_0)/|V_{us}|^2] . \quad (4)$$

Using the OPE representation of  $\delta R_{V+A}^w(s_0)$ , and inputting  $|V_{ud}|$  and the required OPE parameters from other sources, one obtains [8], from Eq. (1),

$$|V_{us}| = \sqrt{\frac{R_{V+A;us}^w(s_0)}{\frac{R_{V+A;ud}^w(s_0)}{|V_{ud}|^2} - [\delta R_{V+A}^w(s_0)]_{OPE}}} . \quad (5)$$

Since, at scales  $s_0 \sim 2 - 3 \text{ GeV}^2$ ,  $[\delta R_{V/A}^w(s_0)]_{OPE}$  is typically much smaller than  $R_{V/A;ud,us}^w(s_0)$  (usually at the few-to-several-% level), Eq. (5) yields a determination of  $|V_{us}|$  with a fractional uncertainty *much* smaller than that on  $[\delta R_{V/A}^w(s_0)]_{OPE}$  itself [8].

A particularly advantageous case, from the point of view of experimental errors, is that based on  $s_0 = m_\tau^2$  and the weight  $w(s) = w_{(00)}(y_\tau) \equiv w_T^{(0,0)}(y_\tau)$ . In this case, the  $us$  and  $ud$  spectral integrals appearing in Eq. (5) are fixed by the total strange and non-strange  $\tau$  branching fractions, allowing one to take advantage of improvements in the errors on a number of the strange branching fractions in advance of the completion of the remeasurement of the full  $us$  spectral distribution. A disadvantage of this approach is that, working with only a single  $s_0$ , one is unable to test the stability of the output  $|V_{us}|$  values with respect to  $s_0$ , a crucial step to ensuring that estimates of the accompanying theoretical uncertainty (which, in some places in the literature, are quoted to be as low as 0.0005) are sufficiently conservative. See below for more on this point.

The OPE representation of  $\delta R_{V/A}^w(s_0)$  is, of necessity, truncated, in both dimension and the perturbative order of the relevant Wilson coefficients. Estimating the associated theoretical uncertainty is complicated by the less-than-ideal convergence of the  $J = 0+1$ ,  $D = 2$  OPE series. Explicitly [27, 28]

$$[\Delta \Pi_\tau(Q^2)]_{D=2}^{OPE} = \frac{3}{2\pi^2} \frac{m_s(Q^2)}{Q^2} \left[ 1 + \frac{7}{3} \bar{a} + 19.93 \bar{a}^2 + 208.75 \bar{a}^3 + \dots \right] , \quad (6)$$

with  $\bar{a} = \alpha_s(Q^2)/\pi$ , and  $\alpha_s(Q^2)$  and  $m_s(Q^2)$  the running coupling and strange quark mass in the  $\overline{MS}$  scheme. Since independent determinations of  $\alpha_s$  [29, 30, 31, 32, 33,

34, 35, 36] imply  $\bar{a}(m_\tau^2) \simeq 0.10$ , the convergence of the  $D = 2$ ,  $J = 0 + 1$  series at the spacelike point on the OPE contour is marginal at best. While (at least if one works with the contour improved (CIPT) prescription [37], in which the large logs are resummed point-by-point along the contour) the convergence of the integrated series can be improved through appropriate weight choices [38], taking into account that  $|\alpha_s(Q^2)|$  decreases as one moves away from the spacelike point along the contour, one expects, in general, rather slow convergence, which makes conventional truncation error estimates potentially unreliable. Fortunately, the growth of  $\alpha_s$  with decreasing  $s_0$  means that omitted higher-order terms become relatively more important at lower scales, and hence that any premature truncation of the slowly converging integrated  $D = 2$ ,  $J = 0 + 1$  series will show up as an unphysical  $s_0$ -dependence in the extracted values of  $|V_{us}|$ . Unphysical  $s_0$ -dependence can also be produced by incorrect input for poorly known, or unknown, condensates relevant to  $D > 4$  OPE contributions ( $D = 6$  and  $8$  in the case of  $w_{(00)}$ ).

Such unphysical  $s_0$ -dependence is, in fact, seen, at a scale significantly larger than the estimated  $D = 2$  truncation error, in recent  $\tau$  decay analyses [11]. This is illustrated in Figure 1, which shows results for  $w_{(00)}$ , and three additional weights,  $w_{10}$ ,  $\hat{w}_{10}$  and  $w_{20}$ , introduced originally to improve the integrated  $D = 2$ ,  $J = 0 + 1$  convergence [38]. Of particular note is the situation for the experimentally favorable  $w_{(00)}$  weight case, where the instability is much larger than full estimated theoretical error.

It is worth noting that the  $D = 2$  truncation component of the 0.0005 theoretical error in the  $w_{(00)}$  case is obtained by combining an uncertainty associated with the residual scale dependence with the shift obtained by omitting the last term included in the  $D = 2$  series, all evaluations being performed using the CIPT prescription and the truncated  $D = 2$  Adler function form. Alternate evaluations, using the truncated correlator (rather than truncated Adler function) form, and/or using the fixed order (FOPT) rather than CIPT prescription, are, however, also possible. At a given, common truncation order, all such evaluations are equivalent to the CIPT Adler function evaluation, differing from it only by corrections of yet-higher order. While the difference of  $|V_{us}|$  values obtained using the  $O(\bar{a}^3)$  and  $O(\bar{a}^4)$  CIPT Adler function evaluations is, indeed, small ( $\delta|V_{us}| = -0.0003$ ), shifting to alternate  $D = 2$  evaluation schemes leads to much larger shifts. For example, shifting from the  $O(\bar{a}^3)$  CIPT Adler function evaluation to the  $O(\bar{a}^4)$  CIPT correlator version yields instead  $\delta|V_{us}| = -0.0008$ , while shifting from the  $O(\bar{a}^4)$  CIPT Adler function (correlator) versions to the  $O(\bar{a}^4)$  FOPT correlator version yields the even larger shifts  $\delta|V_{us}| = 0.0019$  (0.0023) [39]. With plausible arguments in favor of both the CIPT and FOPT prescriptions in the literature [31, 37], such shifts suggest the conventional  $D = 2$  truncation error estimate, which leads to the total estimated theoretical uncertainty,  $\delta|V_{us}| = 0.0005$ , for the  $s_0 = m_\tau^2$ ,  $w_{(00)}$  determination, is far from a conservative one.

In view of the possibility of much-larger-than-anticipated  $D = 2$  truncation uncertainties on the values of  $|V_{us}|$  extracted using the  $\Delta\Pi_\tau$  FESRs, we consider, in what follows, FESRs based on alternate correlator differences designed to have, already at the correlator level, much reduced  $D = 2$  contributions. Such FESRs also allow one to investigate whether the sizeable  $s_0$ -instability observed in the results of the  $w_{(00)}$ -weighted  $\Delta\Pi_\tau$  analysis is a consequence of  $D = 2$  truncation uncertainties, or of unexpectedly

large  $D = 6, 8$  OPE contributions.

### III. NEW MIXED $\tau$ -ELECTROPRODUCTION SUMRULES FOR $|V_{us}|$

Problems associated with the slow convergence of the integrated  $D = 2, J = 0+1$  OPE series can be reduced by considering alternate FESRs based on correlator differences,  $\Delta\Pi$ , sharing with  $\Delta\Pi_\tau$  the vanishing of  $D = 0$  OPE contributions, but having  $D = 2$  contributions suppressed at the correlator level. Since a V/A separation of the flavor  $us$  decay distribution is not presently feasible,  $\Delta\Pi$  should involve the  $us$  V+A combination. The leading order term in the  $D = 2$  Wilson coefficient can be removed by forming the appropriate difference of  $\Pi_{V+A;us}^{(0+1)}$  and the electromagnetic (EM) correlator,  $\Pi_{EM}$ . The following combinations (having the same  $\Pi_{V+A;us}^{(0+1)}$  contribution as  $\Delta\Pi_\tau$ ) have, in addition, vanishing  $D = 0$  contributions:

$$\Delta\Pi_\kappa \equiv 9\Pi_{EM} - \Pi_{V+A;us}^{(0+1)} - 2(2 + \kappa)\Pi_{V;ud}^{(0+1)} + 2\kappa\Pi_{A;ud}^{(0+1)}. \quad (7)$$

The  $\kappa = 1/2$  combination is strictly FB. Bearing in mind that  $\bar{a}(m_\tau^2) \simeq 0.1$ , the corresponding  $D = 2$  OPE contribution,

$$[\Delta\Pi_\kappa(Q^2)]_{D=2}^{OPE} = \frac{3}{2\pi^2} \frac{m_s(Q^2)}{Q^2} \left[ \frac{1}{3}\bar{a} + 4.3839\bar{a}^2 + 44.943\bar{a}^3 + \dots \right] \quad (8)$$

is seen to be strongly suppressed, by more than an order of magnitude, compared to  $[\Delta\Pi_\tau]_{D=2}^{OPE}$ . A similar suppression turns out to be operative for the  $D = 4$  contributions. Explicitly, up to numerically tiny  $O(m_s^4)$  corrections, and neglecting, for simplicity of presentation,  $r = (m_d - m_u)/(m_d + m_u)$ , one has, to  $O(\bar{a}^2)$  [27, 40],

$$[\Delta\Pi_\tau(Q^2)]_{D=4}^{OPE} = \frac{2}{Q^4} [\langle m_\ell \bar{\ell} \ell \rangle - \langle m_s \bar{s} s \rangle] \left( 1 - \bar{a} - \frac{13}{3}\bar{a}^2 \right) \quad (9)$$

$$[\Delta\Pi_\kappa(Q^2)]_{D=4}^{OPE} = \frac{2}{Q^4} \left[ \left( \left( \frac{4 - 16\kappa}{3} \right) \bar{a} + \left( \frac{59 - 236\kappa}{6} \right) \bar{a}^2 \right) \langle m_\ell \bar{\ell} \ell \rangle + \left( \frac{4}{3}\bar{a} + \frac{59}{6}\bar{a}^2 \right) \langle m_s \bar{s} s \rangle \right] \quad (10)$$

where in both cases the strange condensate term is numerically dominant.

Defining  $R_{EM}^w(s_0) = [12\pi^2 S_{EW}/m_\tau^2] \int_0^{s_0} ds w(s) \rho_{EM}(s)$  and  $[\delta R_\kappa^w(s_0)]^{OPE} = [12\pi^2 S_{EW}/m_\tau^2] \oint_{|s|=s_0} ds w(s) [\Delta\Pi_\kappa(s)]^{OPE}$ , one then has, for any analytic  $w(s)$  and any  $s_0$  large enough the OPE representation is reliable, the  $\Delta\Pi_\kappa$  analogue of Eq. (5),

$$|V_{us}| = \sqrt{\frac{R_{V+A;us}^w(s_0)}{9R_{EM}^w(s_0) - \left( \frac{2(2+\kappa)R_{V;ud}^w(s_0) - 2\kappa R_{A;ud}^w(s_0)}{|V_{ud}|^2} \right) - [\Delta R_\kappa^w(s_0)]^{OPE}}}. \quad (11)$$

The suppression of the  $D = 2$  and 4 contributions in  $[\Delta\Pi_\kappa]^{OPE}$  does not persist to higher  $D$ . For example, with  $r_c = \langle\bar{s}s\rangle/\langle\bar{\ell}\ell\rangle$ , the  $D = 6$  contributions, in the vacuum saturation approximation (VSA), become [40]

$$[\Delta\Pi_\tau(Q^2)]_{D=6;VSA}^{OPE} = \frac{\pi\alpha_s\langle\bar{\ell}\ell\rangle^2}{Q^6} \left[ \frac{64}{81} (1 - r_c^2) \right] \quad (12)$$

$$[\Delta\Pi_\kappa(Q^2)]_{D=6;VSA}^{OPE} = \frac{\pi\alpha_s\langle\bar{\ell}\ell\rangle^2}{Q^6} \left[ \left( \frac{-32 + 128\kappa}{9} \right) - \frac{32r_c^2}{9} \right] \quad (13)$$

typically significantly larger for  $\Delta\Pi_\kappa$  than for  $\Delta\Pi_\tau$ .

To deal with such potentially enhanced, but phenomenologically poorly determined,  $D > 4$  contributions, it is useful to employ polynomial weights,  $w(y) = \sum_{m=0} b_m y^m$ , with  $y = s/s_0$ . Integrated  $D = 2k + 2$  OPE contributions then scale as  $1/s_0^k$ . The strong suppression of  $D = 2, 4$  contributions, which scale more slowly with  $s_0$ , then means one can, for example, employ the VSA estimate for  $D = 6$ , and ignore  $D > 6$  contributions, but look for  $s_0$  values large enough that  $|V_{us}|$  becomes stable with respect to  $s_0$ , indicating that  $D > 4$  contributions and/or deviations from the input assumptions about these contributions have decreased to a negligible level.

The expected enhanced role of  $D = 6$  and higher contributions in  $\Delta\Pi_\kappa$  means that higher degree weights like  $w_{10}$ ,  $w_{20}$  and  $\hat{w}_{10}$ , introduced to improve the integrated  $D = 2$  convergence for the  $\Delta\Pi_\tau$  FESRs, are likely to represent less useful choices for the  $\Delta\Pi_\kappa$  analysis. The strong suppression of  $D = 2$  contributions, however, opens up the possibility of using weights which provide less good integrated  $D = 2$  convergence but better control over integrated  $D > 4$  contributions. Thus, e.g., if it is the slow  $D = 2$  convergence which is responsible for the significant  $s_0$ -instability of the  $w_{(00)}$ -weighted  $\Delta\Pi_\tau$  FESR results shown in Figure 1, the analogous  $\Delta\Pi_\kappa$  FESR might be rendered stable by the reduced  $D = 2$  contributions, allowing improvements in the strange branching fractions (whose sum provides an improved determination of  $R_{V+A;us}^{w_{(00)}}(m_\tau^2)$ ) to be used in reducing the error on the numerator in Eq. (11) for  $w = w_{(00)}$  and  $s_0 = m_\tau^2$ . Similarly, it might become possible to employ the weights,  $w_N(y) = 1 - \frac{N}{N-1}y + \frac{y^N}{N-1}$ , which, like  $w_{(00)}$ , display slow integrated  $D = 2$  convergence for  $\Delta\Pi_\tau$  but are useful for handling  $D > 4$  contributions (written generically as  $\sum_{D=6,8,\dots} C_D/Q^D$ ) since (up to corrections of  $O([\alpha_s(m_\tau^2)]^2)$ ) only a single integrated  $D > 4$  contribution,  $(-1)^N C_{2N+2}/[(N-1)s_0^N]$ , survives on the OPE side of the  $w_N$  FESR.

#### IV. INPUT, COMPLICATIONS, RESULTS, AND DISCUSSION

In this section we illustrate the utility of the new mixed FESRs and point out some experimental complications, focussing on the  $\Delta\Pi_{\kappa=1/2}$  case, whose  $D > 4$  contributions vanish in the  $SU(3)_F$  limit, and are thus expected to be optimally suppressed.

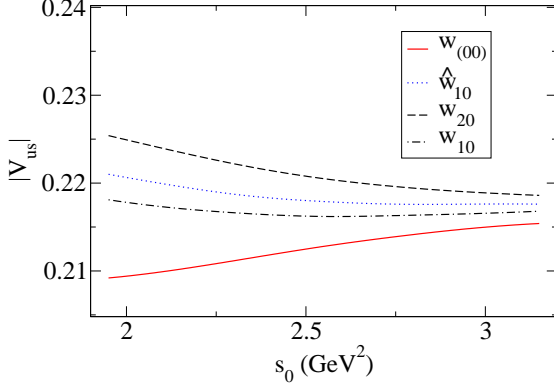


FIG. 1:  $|V_{us}|$  versus  $s_0$  from the  $\Delta\Pi_\tau$  FESRs for  $w_{20}$ ,  $\hat{w}_{10}$ ,  $w_{10}$  and  $w_{(00)}$ .

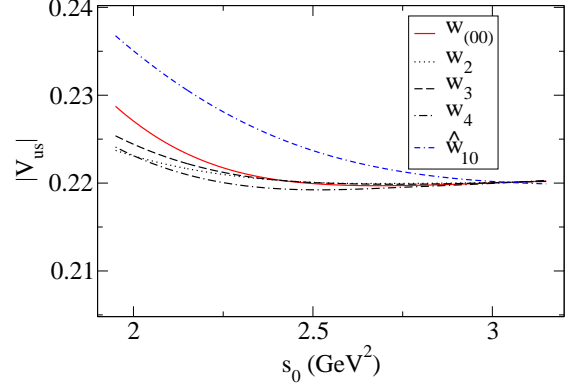


FIG. 2:  $|V_{us}|$  versus  $s_0$  from the  $\Delta\Pi_{\kappa=1/2}$  FESRs for  $w_{(00)}$ ,  $\hat{w}_{10}$ ,  $w_2$ ,  $w_3$  and  $w_4$ .

### A. OPE input

To suppress OPE-breaking contributions from the region of the contour on the RHS of Eq. (1) near the timelike point on the contour, we restrict our attention to  $w(s)$  having a zero of order  $\geq 2$  at  $s = s_0$  and to  $s_0 > 2 \text{ GeV}^2$  [41].

$D = 2$  OPE integrals are evaluated using Eq. (6) and the CIPT prescription [37], with  $\alpha_s(Q^2)$  and  $m_s(Q^2)$  the exact solutions associated with the 4-loop-truncated  $\beta$  and  $\gamma$  functions [42] and the initial conditions,  $m_s(m_\tau^2) = 100 \pm 10 \text{ MeV}$  [10],  $\alpha_s(m_\tau^2) = 0.323(17)$ . The latter is obtained by running a very conservative assessment,  $0.1190(20)$ , of the average of several recent independent  $\alpha_s(M_Z^2)$  determinations [29, 30, 31, 32, 33, 34, 35, 36] down to the  $\tau$  scale using the standard self-consistent combination of 4-loop running and 3-loop matching at the flavor thresholds [43]. To be conservative, we assign the sum of absolute values of the contributions of all computed orders as the truncation component of the  $D = 2$  uncertainty (producing a 100% uncertainty if all contributions have the same sign, larger otherwise). The error on the truncated  $D = 2$  sum associated with that on the overall  $[m_s(m_\tau^2)]^2$  factor is also evaluated using the conservative all-absolute-values prescription. The truncation and  $m_s^2$ -scale errors are combined in quadrature with the much smaller error induced by the uncertainty on  $\alpha_s(m_\tau^2)$  to obtain the full  $D = 2$  error.

$D = 4$  OPE input and uncertainties are as follows.  $\langle m_\ell \bar{\ell} \ell \rangle$  is fixed using the GMOR relation,  $\langle m_s \bar{s} s \rangle$  using conventional ChPT quark mass ratios [44] and the value,  $r_c = \langle \bar{s} s \rangle / \langle \bar{\ell} \ell \rangle = 1.2 \pm 0.3$  obtained by updating the quenched-lattice-data-based determination,  $r_c = 0.8(3)$ , of Ref. [45] using the average,  $f_{B_s}/f_B = 1.21(4)$  [46], of recent  $n_f = 2 + 1$  lattice determinations [47]. The strange condensate term dominates both the  $D = 4$  contribution and its error, but produces only a very small impact on  $|V_{us}|$  as a

consequence of the suppression of the coefficient function seen in Eq. (10).

$D > 4$  contributions involve poorly known or phenomenologically undetermined condensate combinations. We estimate  $D = 6$  contributions using the VSA and ignore  $D \geq 8$  contributions. If  $D > 4$  contributions are small, the details of these assumptions are irrelevant. If not, and the assumptions are inaccurate, the  $1/s_0^2$  ( $1/s_0^3, \dots$ ) dependence of integrated  $D = 6$  ( $8, \dots$ ) contributions will lead to an unphysical  $s_0$ -dependence of  $|V_{us}|$ . We look for weights which produce a good window of  $s_0$ -stability in order to ensure that  $D > 4$  contributions are either negligible or estimated with sufficient accuracy.

## B. Spectral input

Results for  $R_{V/A;ud}^w(s_0)$  and  $R_{V+A;us}^w(s_0)$  are based on the ALEPH  $us$  [19] and  $ud$  [48] spectral data, for which information on the relevant covariance matrices is publicly available. The  $ud$  data has been modified to incorporate the recent improved V/A separation for the  $\bar{K}K\pi$  mode [30] made possible by the BaBar determination of the  $I = 1$   $K\bar{K}\pi$  electroproduction cross-sections [49]. A small rescaling is applied to the continuum  $ud$  V+A distribution to reflect changes in  $S_{EW}$ ,  $R_{V+A;us}$  and the electron branching fraction,  $B_e$ . With the lepton-universality-constrained result  $B_e = 0.17818(32)$  [51] and an updated total strange branching fraction  $B_{us} = 0.02858(71)$ , the  $ud$  normalization is  $R_{V+A;ud} = 3.478(11)$ . For  $|V_{ud}|$ , we use the latest update,  $0.97425(23)$ , from the super-allowed nuclear  $0^+ \rightarrow 0^+$   $\beta$  decay analysis [4].

Though BaBar and Belle have not completed their re-measurements of the inclusive  $us$  distribution,  $dR_{V+A;us}/ds$ , an interim partial update can be obtained (following Ref. [50]) by rescaling the 1999 ALEPH distribution [19], mode-by-mode, by the ratio of new to old branching fractions. The new branching fraction results are taken from Refs. [12, 13, 14, 15, 16, 17, 18]. Unfortunately, this strategy does not allow the corresponding covariance matrix to be updated. The improved precision on the new strange branching fractions can thus be translated into a correspondingly improved  $us$  spectral integral error only for  $w = w_{(00)}$  and  $s_0 = m_\tau^2$ . Since the recently measured  $K$  and  $\pi$  branching fractions [15] are compatible with SM expectations at the  $\sim 2\sigma$  level, we evaluate the  $\pi$  and  $K$  pole spectral integral contributions using the more precisely determined  $\pi_{\mu 2}$  and  $K_{\mu 2}$  input.

$R_{EM}^w(s_0)$  is obtained from the EM spectral function,  $\rho_{EM}(s)$ , which is related to the bare  $e^+e^- \rightarrow \text{hadrons}$  cross-sections,  $\sigma_{bare}(s)$ , by

$$\rho_{EM}(s) = s \sigma_{bare}(s) / 16\pi^3 \alpha_{EM}(0)^2. \quad (14)$$

It is well known that problems exist with the compatibility of the measured  $\pi\pi$  and  $\pi^+\pi^-\pi^0\pi^0$  cross-sections and those implied by  $I = 1$   $\tau$  decay data, even after known isospin-breaking corrections are taken into account [52]. Preliminary BaBar  $\pi^+\pi^-\pi^0\pi^0$  cross-section results [53] reduce considerably the latter discrepancy, but have not yet been finalized. The situation for  $\pi\pi$  is somewhat muddier. The recent KLOE update [54] yields results now in reasonable agreement with CMD2 and SND below the  $\rho$  peak and with a reduced discrepancy above it, while preliminary BaBar results [55] are instead



in better agreement with  $\tau$  expectations. In addition, the most recent  $\tau$ -based analysis [32] produces an  $\alpha_s(M_Z)$  in excellent agreement with two recent high-precision lattice determinations [33], while electroproduction-based analyses (albeit not updated for new post-2005 experimental results, and without the careful fitting of  $D > 4$  OPE contributions performed for the  $\tau$  case) yield values  $\sim 2\sigma$  too low [56], again favoring the  $\tau$  version of the  $I = 1$  spectral distribution. We thus deal with the  $I = 1$  discrepancies by replacing both  $\pi\pi$  and  $4\pi$  EM results with the corresponding  $\tau$  expectations. Since (i) the V/A separation for the  $\bar{K}K\pi$  contribution to  $\tau$  decay has been performed using CVC and the BaBar  $I = 1$  EM cross-sections and (ii) the  $\pi\pi$ ,  $4\pi$  and  $\bar{K}K\pi$  contributions largely saturate  $\rho_{V;ud}^{(0+1)}(s)$  below  $s = m_\tau^2$ , this is effectively equivalent to replacing  $\Delta\Pi_{1/2}$  with the alternate combination

$$\frac{3}{2}\Pi_{V;I=0} - \frac{1}{2}\Pi_{V;ud}^{(0+1)} + \Pi_{A;ud}^{(0+1)} - \Pi_{V+A;us}^{(0+1)} , \quad (15)$$

where  $\Pi_{V;I=0}$  is the  $I = 0$  octet analogue of  $\Pi_{V;ud}^{(0+1)}$ . EM cross-sections are taken from Whalley's 2003 compilation [57] and recent updates reported in Refs. [58, 59, 60]. Where needed, vacuum polarization corrections are computed using F. Jegerlehner's code [61].

### C. Results and discussion

The results for  $|V_{us}|$  as a function of  $s_0$  obtained from the  $\Delta\Pi_{1/2}$  FESRs for  $w_{(00)}$ ,  $w_2$ ,  $w_3$ ,  $w_4$  and the weight,  $\hat{w}_{10}$ , producing the best  $\Delta\Pi_\tau$   $s_0$ -stability plateau in Figure 1 are displayed in Figure 2. In all but the last case a very good  $s_0$ -stability plateau is found. In addition, the  $|V_{us}|$  obtained at the highest accessible scale,  $s_0 = m_\tau^2$  (the right endpoints of the curves) are all, without exception, in extremely good agreement. The very good stability plateau for  $w_{(00)}$  strongly suggests that the instability seen in the analogous  $\Delta\Pi_\tau$  analysis was a result of the slow  $D = 2$  convergence. In contrast, the quality of the stability plateau for  $\hat{w}_{10}$  has deteriorated in going from  $\Delta\Pi_\tau$  to  $\Delta\Pi_{1/2}$ , most likely due to the increased size of  $D > 4$  contributions. Even so, the  $|V_{us}|$  values for  $\hat{w}_{10}$  converge nicely to the stable results from the other weight cases as  $s_0 \rightarrow m_\tau^2$ .

Because of the very good stability found for  $w_{(00)}$ , it is possible to quote a final determination based on  $w = w_{(00)}$  and  $s_0 = m_\tau^2$ , a choice which allows us to benefit from the improved BaBar and Belle strange branching fraction determinations. We find

$$|V_{us}| = 0.2202(27)_{us}(28)_{EM}(2)_{V;ud}(4)_{A;ud}(2)_{OPE} = 0.2202(39) \quad (16)$$

where the errors are those associated with the inclusive  $us$  branching fraction, the residual  $I = 0$  EM spectral integral, the residual inclusive  $ud$  V and  $ud$  A branching fractions, and the combined  $D = 2$  and  $D = 4$  OPE contribution, respectively.

While, within current errors, the result for  $|V_{us}|$  is compatible with either 3-family-unitarity or the recent  $\Delta\Pi_\tau$  determinations, and hence does not help in resolving the  $\sim 3\sigma$  discrepancy between the two, prospects exist for significantly reducing the main components of the error. First, errors on the weighted  $I = 0$  EM integrals will be reduced

through ongoing work on the exclusive EM cross-sections at VEPP2000, BaBar and Belle. Second, errors on the  $us$  spectral integrals will be significantly reduced by BaBar and Belle analyses of both the branching fractions of as-yet-unremeasured strange modes (including the sizable  $\bar{K}^0\pi^0\pi^-$  and previously estimated, but unmeasured,  $\bar{K}3\pi$  and  $\bar{K}4\pi$  modes) and the inclusive  $us$  V+A distribution. Obtaining the inclusive  $us$  distribution, and not just the branching fractions, is crucial to performing the  $s_0$ -stability checks, themselves crucial to demonstrating that  $D = 2$  convergence and  $D > 4$  contributions have, indeed, been brought under good control. To reduce the  $us$ -distribution-induced contribution to the error on  $|V_{us}|$  to, e.g., the  $\sim 0.0005$  level requires  $\sim 1.3 \times 10^{-4}$  precision on the inclusive  $us$  branching fraction, and hence, almost certainly, pursuing previously undetected higher multiplicity modes having branching fractions down to the few  $\times 10^{-5}$  level.

We close by stressing the complementarity of the  $\Delta\Pi_\tau$  and  $\Delta\Pi_{1/2}$  analyses. The latter, by construction, has significantly reduced OPE-induced uncertainties. The smallness of the OPE contributions to the denominator of Eq. (5), however, means that global normalization uncertainties common to the  $ud$  and  $us$  spectral distributions cancel, essentially entirely, in the  $\Delta\Pi_\tau$  determination. This is not the case for the  $\Delta\Pi_{1/2}$  analysis, where EM and  $\tau$  normalization uncertainties are independent, leading to an increased experimental error on  $|V_{us}|$ . As we have seen already in the  $w_{(00)}$  case, employing the same weight in both FESRs and comparing the  $s_0$ -dependences of the resulting  $|V_{us}|$  determinations can also help in shedding light on the source of any  $s_0$ -instabilities found in the  $\Delta\Pi_\tau$  analysis, where OPE-induced errors are more difficult to reliably quantify.

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